Review for midterm:

Material: mostly Chapters 5, 6, 7

ch 5 permutation groups.

$S_n = \{$ all permutations of $\{1, 2, \dots, n\}\}$

$|S_n| = n!$

main results:

- any permutation can be written as a product of disjoint cycles
- if disjoint cycles of perm. π have lengths $m_1, m_2, \dots, m_r$

$\Rightarrow \operatorname{ord}(\pi) = \text{l.c.m.}\{m_1, m_2, \dots, m_r\}$

Example: practice exam: ← cycles NOT disjoint!

$$\pi = (1234)(14)$$

Calculate π^{1802}

Sol. calculate $\mathrm{ord}(\pi)$ first

$(1234)(14) = (1)(234)$ ← product of disjoint cycles

$\Rightarrow$ $\mathrm{ord}(\pi) = 3$

$\Rightarrow$ as $1802 \equiv 2 \mod 3$

it follows $\pi^{1802} = \pi^2 = (234)^2 = (243)$

Problem 2 in practice exam:

how many elements of order 10 in S_7?

If $\pi = c_1 c_2 \ldots c_r$ disjoint cycles, length of $c_i = m_i$

$ord(\pi) = l.c.m.(m_1, m_2, \ldots, m_r) = 10 \Rightarrow$ ① $m_1 + m_2 + \ldots + m_r = 7$
② $l.c.m(m_1, m_2 \ldots m_r) = 10$

obviously in S_7 no 10-cycle $\Rightarrow$ as $5|10$ need a cycle of length 5
$2|10$ " " " " " 2

only possibility: one 5-cycle, one 2-cycle } disjoint

how many 5-cycles in S_7?

we have $\binom{7}{5} = \binom{7}{2} = 21$ subsets with 5 elements of $\{1, 2, \ldots 7\}$

pick one of those subsets, say $\{1, 2, 3, 4, 5\}$

how many 5-cycles in S_5?

Solution: a 5-cycle in S_5 given by writing the numbers 1, 2, 3, 4, 5 in random order (e.g. $(3\,5\,2\,4\,1)$)

have $5!$ possibilities.

⚠ Observe: applying a cyclic permutation to it we still get same permutation!

e.g. $(35241) = (52413) = (24135)$

$\Rightarrow$ have $\dfrac{5!}{5} = 4!$ different 5-cycles in S_5

have seen: have $\binom{7}{5} = 21$ subsets of $\{1,2,\dots,7\}$ with 5 elements

$\Rightarrow$ have alltogether $21 \cdot 4! = 21 \cdot 24$ 5-cycles in S_7

This is essentially the proof of the more general

Theorem: let $m \leq n$.

Then we have $\boxed{\binom{n}{m} \cdot (m-1)! \text{ m-cycles in } S_n.}$

e.g. $n=7$ $m=5$: $\binom{7}{5} \cdot 4! = 21 \cdot 24$

$n=4$ $m=3$ $\Rightarrow$ have $\binom{4}{3} \cdot 2! = 4 \cdot 2 = 8$

3-cycles in S_4

(checked explicitly before)

Solution of problem:

have seen: $21 \cdot 24 = {}^{48}_{24} = 504$ 5-cycles in S_7.

after fixing 5-cycle only 2 numbers left for 2-cycle.

only one possibility for 2-cycle.

Solution: 504

othe results : odd / even permutations

subgroup A_n of even permutation

$|A_n| = \frac{n!}{2}$ elements.

Chapter 6 Isomorphism :

An isom. $\Phi : G \to H$ is a 1-1 and onto map

satisfying $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in G$

Important Problem :

Given two groups G and H, decide whether they are isomorphic or not

General Strategy:

to prove $G \cong H$:

- construct an isom. $\Phi: G \to H$
- have already shown:

  if G, H both cyclic with $|G| = |H|$

  $\Rightarrow G \cong H$

  if $G = \langle a \rangle$, $H = \langle b \rangle$

  isom. given by $a^i \mapsto b^i$, $i = 0, 1, \dots |G|-1$.

to prove $G \not\cong H$ show that some of prop. of an isom. are violated

e.g. • $|G| \neq |H| \Rightarrow G \not\cong H$

- know: $\operatorname{ord} \Phi(a) = \operatorname{ord} a$

  if we can find k s.t. $\#\{a \in G, \operatorname{ord}(a) = k\} \neq \#\{b \in H, \operatorname{ord}(b) = k\}$

  $\Rightarrow G \not\cong H$

example: $H = \{ id, (12)(34), (13)(24), (14)(23) \}$

subgroup of $A_4 < S_4$.

H has 3 elements of order 2

$\Rightarrow H \not\cong \mathbb{Z}_4$ $\quad$ $\mathbb{Z}_4$ only has one element of order 2, namely 2.

in practice exam:

Is $U(5) \overset{?}{\cong} U(10)$?

$U(5) = \{1, 2, 3, 4\}$ $\quad$ $|U(5)| = |U(10)|$

$U(10) = \{1, 3, 7, 9\}$

$U(5)$ cyclic: $U(5) = \langle 2 \rangle$ $\quad$ $(2^2 = 4, 2^3 = 8 = 3 \bmod 5, 2^4 = 16 = 1 \bmod 5)$

$U(10)$ " : $U(10) = \langle 3 \rangle$ $\quad$ $(3^2 = 9, 3^3 = 27 = 7 \bmod 10, 3^4 = 81 = 1 \bmod 10)$

$\Rightarrow$ both $U(5)$ and $U(10)$ are cyclic groups of order 4

$\Rightarrow$ $U(5) \cong U(10)$

isom. given by $2^j \in U(5) \to 3^j \in U(10)$

$j = 0, 1, 2, 3$

automorphisms

$\alpha : G \to G$ where α is an isom.

have seen: If $G = \mathbb{Z}_n$

we have $\phi(n) = \#\{j, 0 < j < n, \gcd(j,n) = 1\}$

$\phi(n)$ automorphisms of $\mathbb{Z}_n$

If $j \in U(n)$ can define autom. $\alpha_j : \mathbb{Z}_n \to \mathbb{Z}_n$

$k \to jk$.

$(\phi(n) = |U(n)|)$